Indian Statistical Institute
II Semestral Examination 2008-2009
M.Math. II year
Partial Deferential Equations

Date:00-00-2009 Duration: 3 Hours Max Marks you can get is 60

1. Let G be any open subset of R^n, $m = 1, 2, 3, , , 1 \leq p < \infty$. Show that $C^\infty(G) \cap W^{m,p}(G)$ is dense in $W^{m,p}(G)$. [6]

2. Let H be a Hilbert space over the field F (of course $F = R$ or $F = C$). Let $a : H \times H \longrightarrow F$ be continuous, linear in the first variable and conjugate linear in the second variable. Assume that there exists a constant $k > o$ such that
$$a(u, u) \geq k \parallel u \parallel^2$$
for all u in H. Let $b : H \longrightarrow F$ be linear continuous map. Show that there exists a unique x_o in H such that $a(u, x_0) = b(u)$ for all u in H.

[5]

3. Prove Poincare inequality for the bounded open set $(a_1, b_1) \times (a_2, b_2)$ of R^2. [3]

4. Define $P(x, y) = \frac{1-x^2}{|x-y|^n}$ for $\mid x \mid < 1 = \mid y \mid$, $x, y \varepsilon R^n$. Show that $\triangle_x P(x, y) = 0$. [3]

5. LetΩ be any bounded open subset of R^n with smooth boundary $\partial\Omega$. Prove Greens representation theorem:

   Let $u \varepsilon C^2(R^n)$. Then for $y \varepsilon\ \Omega,\ u(y)$
   $= \int\limits_{\partial\Omega} d\ \sigma(x)\ \{u(x)\frac{\partial\lceil}{\partial\vartheta_x}(x, y) - \lceil(x, y)\frac{\partial u}{\partial\vartheta}(x)\}$
   $+ \int\limits_{\Omega} dx \lceil(x, y)(\triangle u)(x)$

   Here σ is the surface area measure of $\partial\Omega, \vartheta$ is the unit outer normal for $\partial\Omega$. $\lceil$ is given by the formula $\lceil(x, y) = \frac{1}{2\pi} \log \mid x - y \mid$ for x, y in R^2 $\lceil(x, y) = \frac{1}{(2-n)W_n}\ |x - y|^{2-n}$ for $x, y \varepsilon R^n$ $n = 3, 4, ...$

   You can assume that $\lceil$ is a fundamental solution for the Laplacian $\triangle$.

   [10]

6. a) Let $f : R \times R \longrightarrow R$ be any C^2 function,$f = f(x_1, x_2)$. Define $g : R \longrightarrow R$ by $g(t) = \int_o^t$ ds $f(t - s, s)$. Then show that

   $g^1(t) = f(0, t) + \int_o^t$ ds $\frac{\partial f}{\partial x_1}(t - s, s)$. [2]

b) Prove Duhamels principle. Let $W : R^n \times R \longrightarrow \mathbb{R}$ be any C^2 function, $W = W(x,t)$, x in R^n, t in R. For each $s > o$, let $v(x,t,s)$ be a solution of

$\frac{\partial^2}{\partial t^2} v(x,t,s) = \triangle_x v(x,t,s)$

$v(x,o,s) = 0$

$\frac{\partial v}{\partial t}(x,0,s) = W(x,s)$

Here $x \varepsilon R^n, t \varepsilon R$. Define $u : R^n \times R \longrightarrow R$. by $u(x,t) = \int_0^t$ ds $v(x,t-s,s)$ for $t \geq 0$. Then show that u satisfies $u_{tt} - \triangle_x u = W$ on $R^n \times (0,\infty)$, $u(x,0) = 0 = u_t(x,0)$. [4]

7. Let $f : R^n \longrightarrow R$ be a C^∞ function. Define the spherical mean for f by

   $M_f(x,r) = \frac{1}{W_n} \int\limits_{|y|=1} f(x+ry) d\sigma(y).$

   where d $\sigma(y)$ is the surface measure on $S^{n-1} = \{y \varepsilon R^n : |y| = 1\}$ normalized so that $\sigma(S^{n-1}) = W_n$. Prove the Darboux equation.

   $\{\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\}\ M_f(x,r) = \triangle_x M_f(x,r) = M_{\triangle f}(x,r).$ [8]

8. Let $f \varepsilon \mathbf{S}[R^d]$. For all $u \varepsilon \mathbf{S}[R^d]$. Show that $\parallel fu \parallel_{H^s} \leq C(s,f) \parallel u \parallel_{H^s}$ where $C(s,f)$ is a constant depending on f. [5]

9. Let p be a homogeneous elliptic polynomial of degree N. Define

   $b(y) = \frac{|y|^N}{1+|y|^N} . \frac{1}{p(y)}$ for $y \neq 0$. Define B by $(Bu)\hat{}(\xi) = b(\xi)\hat{u}(\xi)$. Show that $\parallel Bu \parallel_{H^s} \leq C \parallel u \parallel_{H^{s+N}}$ where C is a constant independent of u

   [3]

10. Define $K(t,x) = \chi_{(0,\infty)}(t) t^{-\frac{n}{2}} exp\ [-(\frac{x^2}{4t})]$ Show that λK is a fundamental solution for $\frac{\partial}{\partial t}\triangle_x$ for a suitable constant λ. [15]